Seminar I
Boolean and Modal Algebras

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Gödel gave us two translations: (1) classical into intuitionistic using not-not, and (2) intuitionistic into S4-modal logic.

Tarski and McKinsey reviewed all this algebraically in propositional logic, proving completeness of (2).

Mostowski suggested the algebraic interpretation of quantifiers.

Rasiowa and Sikorski went further with first-order logic, giving many completeness proofs (**pace** Kanger, Hintikka and Kripke).

Montague applied higher-order modal logic to linguistics.

Solovay and Scott showed how Cohen's forcing for ZFC can be considered under (1). Bell wrote a book (now 3rd ed.).

Gallin studied a Boolean-valued version of Montague semantics.

Myhill, Goodman, Flagg and Scedrov made proposals about modal ZF.

Fitting studied modal ZF models and he and Smullyan worked out forcing results using both (1) and (2).
What is a Lattice?

0 ≤ x ≤ 1  

Bounded

x ≤ x  

Partially Ordered Set

x ≤ y & y ≤ z  \implies  x ≤ z

x ≤ y & y ≤ x  \implies  x = y

With sups

x \lor y ≤ z  \iff  x ≤ z & y ≤ z

&

With infs

z ≤ x \land y  \iff  z ≤ x & z ≤ y
What is a Complete Lattice?

\[ \bigvee_{i \in I} x_i \leq y \iff (\forall i \in I) \ x_i \leq y \]

\[ y \leq \bigwedge_{i \in I} x_i \iff (\forall i \in I) \ y \leq x_i \]

**Note:**

\[ \bigwedge_{i \in I} x_i = \bigvee \{y \mid (\forall i \in I) \ y \leq x_i \} \]
What is a Heyting Algebra?

\[ x \leq y \rightarrow z \iff x \land y \leq z \]

What is a Boolean Algebra?

\[ x \leq (y \rightarrow z) \lor w \iff x \land y \leq z \lor w \]

Alternatively using Negation

\[ x \leq \neg y \lor z \iff x \land y \leq z \]
Theorem: Every Heyting algebra is **distributive**:

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]

Theorem: Every complete Heyting algebra is **(\land \lor)**-**distributive**:

\[ x \land \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \land y_i) \]

Note: The dual law does not follow for **complete Heyting algebras**.
**Proof Outline**

To show that

\[ x \land (y \lor z) \leq (x \land y) \lor (x \land z), \]

it is sufficient to show that

\[ y \lor z \leq x \rightarrow ((x \land y) \lor (x \land z)). \]

So, it is sufficient to show that

\[ y \leq x \rightarrow ((x \land y) \lor (x \land z)), \]

(and also for \( z \)). But this last comes down to

\[ x \land y \leq (x \land y) \lor (x \land z). \]

Now reverse the argument. Q.E.D.

**Note:** The \((\land \lor)\)-law has a similar proof.
First-Order Algebraic Semantics

\[ aRb \] = given

\[ \Phi \land \Psi ] = [ \Phi ] \land [ \Psi ] \]

\[ \Phi \lor \Psi ] = [ \Phi ] \lor [ \Psi ] \]

\[ \Phi \rightarrow \Psi ] = [ \Phi ] \rightarrow [ \Psi ] \]

\[ \exists x. \Phi (x) ] = \bigvee _{a \in A} [ \Phi (a) ] \]

\[ \forall x. \Phi (x) ] = \bigwedge _{a \in A} [ \Phi (a) ] \]

Note: A number of details are being ignored here.
Semantical Completeness

A sentence $\Phi$ is provable in intuitionistic first-order logic if, and only if,

$$\lfloor \Phi \rfloor = 1$$

whatever the interpretation in a complete Heyting algebra.

Note: The proof from left to right is obvious! And the result holds for classical and modal logic.
Generic Completeness

There is – relative to the choice of language – a single algebra such that

\[ \phi \]

if \( \llbracket \phi \rrbracket = 1 \) for this algebra,

then \( \phi \) is provable in intuitionistic (classical) (modal) first-order logic.

The proof goes through the Lindenbaum algebra and the MacNeille completion of lattices.
MacNeille Completion I.

The completion embeds a lattice into the lattice of those ideals that are equal to the lower bounds of all their upper bounds.

**Hint**: Think of Dedekind cuts.

**The Good**: The completion preserves all the *existing* sups and infs.

**The Bad**: The MacNeille completion of a distributive lattice is *not always* distributive!
MacNeille Completion II.

• There are many (equational) varieties between Heyting and Boolean algebras.

• However, the completeness process only puts us in the same variety in the two extreme cases.

• But, it does work for the extension to modal S4 Heyting and Boolean algebras (to be explained next).
What Happened to Gödel?

The usual \( \{0,1\} \)-valued completeness theorem follows from the Boolean version for countable languages via the Rasiowa-Sikorski Lemma: Ultrafilters can be found preserving any given countable list of sups and infs in a Boolean algebra.

Hence, the MacNeille completion is not needed for \( \{0,1\} \)-valued completeness.
What is a Lewis (S4) Algebra?

A complete Boolean algebra plus a “necessity” operator satisfying:

\[ \Box 1 = 1 \]

\[ \Box \Box p = \Box p \leq p \]

\[ \Box (p \land q) = \Box p \land \Box q \]

Note: The second two laws can be combined:

\[ \Box p = \bigvee \{q \mid q = \Box q \leq p\}. \]

“Possibility” is defined as \[ \Diamond p = \neg \neg \neg p. \]
Some Abbreviations

Ha = Heyting Algebra
cHa = Complete Heyting Algebra
Ba = Boolean Algebra
cBa = Complete Boolean Algebra
La = Lewis Algebra
cLa = Complete Lewis Algebra

Note: For semantics don’t forget to add:

$[\square \Phi] = \square [\Phi]$
What is a Frame?

Definition. A frame is any complete lattice which is \((\wedge \vee)\)-distributive.

Theorem. In a cLa, the \(\Box\)-stable elements form a subframe.

Theorem. In a cBa, any subframe creates a cLa.

Hint: We can define: \(\Box p = \vee\{q \in H | q \leq p\}\), where \(H\) is the subframe. Such structures can be regarded as abstract topological spaces.
Theorem. Every frame can be made into a cHa.

Define: $q \rightarrow r = \bigvee \{p | p \land q \leq r\}$.

Corollary. In a cHa every subframe can be regarded as a cHa (but not with the same $\rightarrow$).

Note: $\neg p = p \rightarrow 0$. 

An Important Theorem
Boole vs. Heyting vs. Lewis

**Theorem (old).** For every cBa $B$, the cHa $H$ of all ideals of $B$ is such that $B \cong \{ \neg \neg p \mid p \in H \}$.

**Theorem (new?).** For every cLa $L$, the cHa $H$ of all ideals of $L$ is such that $L \cong \{ \neg \neg p \mid p \in H \}$ and $\square_L p = \neg \neg \square_H p$, where we define $\square_H p = \{ q \in L \mid \exists r \in p[q \leq \square_L r] \}$.

**Theorem (old).** For every cHa $H$, there is a (non-canonical) cLa $L$ such that $H \cong \{ \square p \mid p \in L \}$. 
Question. What are Lewis (S5) algebras?

Answer. A cBa and a complete Boolean subalgebra.

Hint: The extra (S5) axiom amounts to

\[ \Diamond p = \Box \Diamond p. \]

By way of example, think of a powerset and the subalgebra of sets invariant under an equivalence relation.
A Very Brief Bibliography

Books


Articles

Melvin Fitting, Intensional Logic—Beyond First Order, in: [TIL], pp 87--108.


Michael P. Fourman, and Dana S. Scott, Sheaves and logic, In: [AOS], pp. 302–401.


Andrej Scedrov, *Extending Gödel’s modal interpretation to type theory and set theory*, in: [IMA], pp. 11-46.


Dana Scott and Peter Krauss, *Assigning probabilities to logical formulas*, in: [AIL], pp. 219-264.

