Counterfactual Logic and the Necessity of Mathematics

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Abstract

This paper is concerned with counterfactual logic and its implications for the modal status of mathematical claims. It is most directly a response to an ambitious program by Yli-Vakkuri and Hawthorne (2018), who seek to establish that mathematics is committed to its own necessity. I claim that their argument fails to establish this result for two reasons. First, their assumptions force our hand on a controversial debate within counterfactual logic. In particular, they license counterfactual strengthening—the inference from 'If A were true then C would be true' to 'If A and B were true then C would be true'—which many reject. Second, the system they develop is provably equivalent to appending Deduction Theorem to a **T** modal logic. It is unsurprising that the combination of Deduction Theorem with **T** results in necessitation; indeed, it is precisely for this reason that many logicians reject Deduction Theorem in modal contexts. If Deduction Theorem is unacceptable for modal logic, it cannot be employed to derive the necessity of mathematics.

Introduction

Mathematical truths necessarily obtain. While it is possible for Hillary Clinton to have won the 2016 presidential election, it is necessary that $2 + 2 = 4$; while the Axis powers could have won World War II, it could not be that negative numbers have real square roots; and while there are some possible worlds in which there are an even number of stars, there are none in which all Fermat numbers are prime. History might have progressed far differently than it actually did, and the laws of physics might even have diverged wildly from what they actually are, but, the received wisdom goes, pure mathematics concerns what is necessarily true—it may even be the paradigmatic example of a realm of necessary truths.

This much is uncontroversial (or, at least, as uncontroversial as anything ever is in philosophy), but there is currently no consensus on the foundations for the necessity of mathematics. What is it in virtue of that these truths, rather than others, hold necessarily? Are we justified in our collective confidence that they could not have been otherwise? Is there a division of labor, such that mathematicians provide the truths and philosophers the necessity, or is mathematics itself committed to the necessity of its claims?

Numerous proposals are available in the literature. According to one, the necessity of mathematics is secured by the strength of our intuitions.[1] Perhaps conceivability is a guide to possibility; the fact that it is conceivable that p is evidence that it is possible that p, and the fact that it is inconceivable that p is evidence that it is impossible that p. If this is so, then our inability to conceive of a way for two and two to make five is evidence that it is impossible for two and two to make five. And if all mathematical falsehoods are similarly inconceivable, we can be confident in the necessity of mathematical truths. Of course, this strategy does not determine the metaphysical basis for the necessity of mathematics, but it could explain why our belief in this necessity is justified. Alternatively, according to neologicists—who maintain that arithmetic is reducible to logic—the necessity of mathematics results from the necessity of logic.[2] Arguably, the necessity of logic is as reasonable a starting-point as any in modal inquiry, so if arithmetic is reducible to logic, then logical truths generate arithmetic truths that necessarily obtain. However, following the results of Gödel's incompleteness theorem, neologicists typically aim only to establish the necessity of a fragment of mathematics.[3] Still others argue that we ought not be nearly so confident in the necessity of mathematics as we currently are.[4] Mathematicians are standardly content to prove that something is true; they seldom bother to prove that it is necessarily true. Indeed, terms like 'necessity' and 'possibility' are conspicuously absent from the vast majority of mathematical texts. Philosophers, some claim, step in when mathematicians' work is complete and (perhaps erroneously) attribute necessity to the results of their theorems.

Recently, Yli-Vakkuri and Hawthorne (2018) provide a novel defense for the necessity of mathematics. They argue that counterfactual logic and mathematical practice jointly

[1]See Bealer (2002). For a more general discussion of the connection between conceivability and possibility (especially in light of the Kripke (1980) development of the necessary *a posteriori*) see Gendler and Hawthorne (2002).

[2]See, e.g., Hale and Wright (2001).

[3]Yli-Vakkuri and Hawthorne, for example, claim "The neologicist strategy has inherent limitations. It can, at best, establish only the necessity of those mathematical truths that are provable in whatever axiomatic system it uses. By Gödel's first incompleteness theorem, we know that these cannot even include all truths of first-order arithmetic" (pg. 4). For my part, I find this modesty premature. It is worth recalling, as philosophers are often prone to forget, that arithmetic is only incomplete on the assumption that its axioms ought to be decidable—i.e., that an infinitely large computer with an infinite amount of time ought to be able to determine whether a given formula is an axiom. There are numerous complete, albeit undecidable, axiomatizations of arithmetic. Whether decidability is an appropriate restriction depends largely on our theoretical aims. I see no reason why axioms ought to be decidable when the subject is the reduction of arithmetic to logic; all that is required is that each axiom be a principle of logic. For example, the ω-rule, according to which one may infer $\forall x Fx$ after infinitely many steps determining that $Fa, Fb, ...$ is undecidable but arguably a principle of logic (minimally, it seems as plausibly a principle of logic as Hume's Principle, according to which the number of Fs = the number of Gs just in case there is a one-to-one correspondence between the Fs and Gs, something neologicists often assume). I suspect that this humility arises because neologicists are typically committed not only to the reduction of arithmetic to logic in general, but to Frege (1884)'s derivation in particular. This strategy inevitably inherits the incompleteness of Peano arithmetic.

[4]See, e.g., Hodges (Forthcoming).

entail that mathematics is committed to its own necessity: that, for any sentence *S* within the language of pure mathematics, if *S* is true then *S* is necessarily true. Their assumptions do not merely entail that mathematics is committed to the necessity of its claims, but to an **S5** modal logic in particular. Its modal commitments run deep.

I admit that when I first encountered this paper, I was captivated by its result. It seemed to me that—at long last—we had no need to rely on the strength of intuition or the dubious program of neologicism. A rigorous derivation could take their place. Perfectly innocuous assumptions about counterfactual logic entail that mathematics is committed to its own necessity. Indeed, I suspected that this would eventually be seen as one of the most significant contributions to the philosophy of mathematics in many years.

As time progressed, my doubts developed. I no longer believe that this program succeeds. This paper principally consists of two worries for Yli-Vakkuri and Hawthorne's argument and its relation to the logical system they develop. In my mind, these worries are simply that: worries. They are troubling enough to undermine my confidence in this program's success—they do not ensure its failure. Nevertheless, I suspect that much would need to be done to restore confidence in their result. The first problem I raise is that their assumptions entail the success of counterfactual strengthening—the inference from 'If *A* were true then *C* would be true' to 'If *A* and *B* were true then *C* would be true.' Many deny the felicity of counterfactual strengthening in ordinary modal contexts. Indeed, the Stalnaker (1968)/Lewis (1973*a*) semantics for counterfactual conditionals, which remains dominant in the discipline at large, entails that counterfactual strengthening fails. Whether Yli-Vakkuri and Hawthorne's assumptions are tenable depends (at least partially) on whether the problematic implications of strengthening can be derived in the language of pure mathematics. This requires a more precise account of what constitutes pure mathematics than is currently available. The second problem is than an axiom (Deduction Theorem) which receives no substantive defense is largely—indeed, almost solely—responsible for their paper's conclusion. This axiom is extraordinarily controversial; there are many who deny that it obtains for standard modal logics. Without a reason to accept that this axiom is true, we remain without a compelling argument for the necessity of mathematics.

Before turning to the details of Yli-Vakkuri and Hawthorne's account, a brief note on a tension between the worries I raise. While the first might reasonably be interpreted as the claim that their assumptions are far too strong (in that they force our hand on longstanding and intractable debate about counterfactual logic), the second is that the bulk of these very same assumptions are too weak to secure any theoretically interesting results (indeed, hardly more than are required to ensure that the language of mathematics is capable of expressing any modal claims at all). I will attempt to alleviate this tension in some concluding remarks; for the moment I simply note that it exists.

The Necessity of Mathematics

Yli-Vakkuri and Hawthorne's program fits broadly within a reorientation occurring in metaphysics. Following the formalization of modal logic in the 1960's, and the apparent theoretical uses for modality that ensued, many took the notions of possibility or necessity to be primitive, and defined other modal notions (such as the counterfactual conditional) in terms of them. In contrast, some contemporary philosophers maintain that the counterfactual conditional ought to be taken as primitive, and necessity and possibility defined in terms of it.[5] The crucial definition of necessity in terms of counterfactuality is the following:

$$\Box A =_{df} \neg A \,\Box\!\!\rightarrow \bot$$

The claim that it is necessary that A amounts to the claim that if A were false, then the absurd would obtain. This definition receives support on several fronts. It is an immediate consequence of the aforementioned Stalnaker/Lewis semantics for counterfactual conditionals, according to which sentences of the form 'If A were true then B would be true' hold just in case the closest possible worlds in which A is true are also possible worlds in which B is true.[6] But perhaps the most compelling defense of this principle occurs in Williamson (2007), who demonstrates that it follows from a **K** modal logic—the weakest modal logic standardly available—and the following two principles:

NECESSITY: $\Box(A \rightarrow B) \rightarrow (A \,\Box\!\!\rightarrow B)$

POSSIBILITY: $(A \,\Box\!\!\rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

These assert, respectively, that if it is necessary that if A then B, then if A were to obtain then B would obtain, and that if it is the case that if A were to obtain then B would obtain, then if it is possible that A then it is possible that B. With the counterfactual definition of necessity at hand, possibility can be defined in the standard way:

[5] See Williamson (2007). This trend is in its infancy; it remains to be seen whether it will stand the test of time. Part of the motivation for this approach is that, Williamson maintains, we have more direct epistemic access to counterfactual conditionals than we have to necessity and possibility. While scientific experiments may inform us of what would happen if electrons were to pass through an open slit, it is not obvious that they inform us that water is necessarily H_2O. However, for alternate accounts of our epistemic access to modality, see, e.g., Hale (2003); Lowe (2012); Kment (2018).

[6] This is a rough gloss on their views, which differ in philosophically important ways. In particular, Stalnaker's similarity relation selects a unique, most similar w' for each possible world w, and determines the truth of counterfactuals by what occurs in it. Lewis, in contrast, evaluates counterfactuals by truth at the closest possible worlds (plural) and does not assume that there is a unique most-similar world. Each version has benefits over the other. For example, it is a consequence of Lewis's—but not Stalnaker's—view that the Counterfactual Excluded Middle ($A \,\Box\!\!\rightarrow B \vee A \,\Box\!\!\rightarrow \neg B$) fails. I take it that debates, important though they are, have no bearing on the current project.

$$\Diamond A =_{df} \neg\Box\neg A$$

With an eye toward the necessity of mathematics, Yli-Vakkuri and Hawthorne appeal to counterfactual conditionals occurring in mathematical texts. Sentences like "[If] there were a machine computing t [then] it would have some number k of states" (Boolos, Burgess and Jeffrey (2007)) regularly appear, and are naturally interpreted as counterfactual conditionals. Given that the truth-values of these sentences depend upon merely possible situations, mathematics is plausibly committed to a wide modal scope.

There is a natural objection to this interpretation which ought to be set aside. Arguably, counterfactual conditionals with necessary or impossible antecedents are somehow defective. A counterfactual with a necessary antecedent may collapse into the material conditional (because the closest world in which the antecedent obtains is the actual world), and a counterfactual with an impossible antecedent may be ill-formed (because there are no worlds in which the antecedent obtains).[7] Given the charitable assumption that mathematicians' assertions are neither trivial nor ill-formed, some might reasonably prefer alternate interpretations of Boolos, Burgess and Jeffrey's sorts of claims. However, it is worth recalling that Yli-Vakkuri and Hawthorne's imagined interlocutors are those who maintain that mathematical truths are contingent; they cannot object by appealing to the inadmissibility of counterfactuals with necessary or impossible antecedents, because they do not believe that mathematical counterfactuals *have* necessary or impossible antecedents.

Yli-Vakkuri and Hawthorne assume that the language of pure mathematics is at least equipped with sentences (which are denoted by 'A,' 'B,' etc. for individual sentences and by 'Γ,' Π,' etc. for collections of sentences), the classical logical connectives, the counterfactual connective $\Box\!\!\rightarrow$, the absurdity operator $\bot$ and a symbol for informal provability $\vdash$. The least familiar of these is, presumably, the notion of informal provability. Informal proofs are mathematically rigorous; the main difference between informal and formal proofs is that the results of informal proofs are universally true, while falsehoods are formally provable in systems with false axioms. Additionally, the notion of informal provability is sensitive to mathematical practice: the fact that mathematicians regularly license a particular kind of inference is evidence that it is admissible in informal proofs.

In addition to the counterfactual definition of necessity, Hawthorne and Yli-Vakkuri make the following assumptions:

CLASSICAL CONSEQUENCE	$\Gamma \vdash A$ whenever A follows from Γ by classical logic.
MODUS PONENS	$\Gamma, A \Rightarrow B, A \vdash B$ where $\Rightarrow$ is either the

[7]This is the standard Stalnaker/Lewis line. There has, however, been a sustained defense of counterpossibles: counterfactual conditionals with impossible antecedents. See, for example, Cohen (1987); Mares (1997); Goodman (2004); Bjerring (2013); Brogaard and Salerno (2013). Nevertheless, I note that Yli-Vakkuri and Hawthorne do not avoid the collapse of the counterfactual conditional into the material conditional. As I mention below, it is provable on their assumptions that $A \Box\!\!\rightarrow B$ iff $A \rightarrow B$.

	counterfactual or material conditional
CUT	If $\Gamma \vdash A_1, ..., A_n$ and $\Pi, A_1, ..., A_n \vdash B$ then $\Pi, \Gamma \vdash B$
COUNTERFACTUAL DEDUCTION	If $\Gamma, A \vdash B$, then $\Gamma \vdash A \mathrel{\Box\!\!\rightarrow} B$
DEDUCTION THEOREM	If $\Gamma, A \vdash B$, then $\Gamma \vdash A \rightarrow B$

Classical Consequence, Modus Ponens, Cut and Deduction Theorem are all, they claim, uncontroversial. The novel assumption is Counterfactual Deduction. But there is plenty of textual evidence that mathematicians assume that it is true. Take, for example:

> Let us designate the set of all such Gödel numbers by R, and let us suppose that R is recursively enumerable. Then, since $R \neq \varnothing$, there would exist a recursive function $f(n)$ whose range is R. (Davis, 1958, pg. 78)

Davis recognizes that, under the assumption that R is recursively enumerable, it is provable that there is a function whose range is R. What he concludes, then, is a counterfactual: if R were recursively enumerable, then there would be a function with R as its range. This is an instance of Counterfactual Deduction.

Or consider an elementary proof that there are infinitely many prime numbers. Suppose, for reductio, that there were finitely many primes. In this case, these primes would have a product n. The number $n + 1$ would not be evenly divisible by any prime number (except the number 1, depending on whether 1 regarded as prime), and would therefore be prime. However, $n + 1$ is not a factor of n, because it is larger than n. Therefore, n would not be the product of all primes, which contradicts the former claim that it is the product of all primes.

Several counterfactuals occurred in this proof. The relevant inference occurs from what is *provable* from the claim that there are finitely many primes to what *would occur* were there finitely many primes. This too is an instance of Counterfactual Deduction. Notably, the other principles Yli-Vakkuri and Hawthorne rely upon receive no sustained defense or discussion.

With such principles at hand, the derivation of the necessity of mathematics is as follows. Let A be an arbitrary sentence in the language of mathematics. From Classical Consequence, we have:

$$A, \neg A \vdash \bot$$

Counterfactual Deduction then entails:

$$A \vdash \neg A \mathrel{\Box\!\!\rightarrow} \bot$$

The counterfactual definition of necessity then gives us:

$$A \vdash \Box A$$

Deduction Theorem then entails:

$$\varnothing \vdash A \to \Box A$$

This does not simply assert that if a sentence is true then it is necessarily true; it makes the stronger claim that it is *provable* that if A is true then it is necessarily true.[8]

Replacing A with $\Box A$ and $\Diamond A$ results in:

4: $\vdash \Box A \to \Box\Box A$

5: $\vdash \Diamond A \to \Box\Diamond A$

Classical Consequence, Deduction Theorem, Modus Ponens and Cut collectively imply that:

$$(\neg A \mathrel{\Box\!\!\to} \bot) \to A$$

From the counterfactual definition of necessity, we then have:

T: $\Box A \to A$

Additionally, the **K** axioms of $\Box(A \to B) \to (\Box A \to \Box B)$ and $\vdash A \to \vdash \Box A$ are both theorems. These suffice to axiomatize **S5** modal logic. And so, Yli-Vakkuri and Hawthorne conclude, mathematics is committed not only to its own necessity, but to an **S5** system in particular. Far from being agnostic about its modal commitments, mathematics determines the system of modal logic which governs its theorems' results.

A Worry Concerning Counterfactual Strengthening

It is my hope that the previous (admittedly somewhat cursory) overview conveys both the structure and initial appeal of Yli-Vakkuri and Hawthorne's argument. This argu-

[8]Note that this need not conflict with the incompleteness of various mathematical systems. There may be many sentences A in the language of pure mathematics such that A is true but $\vdash A$ is false. What this asserts is that, even in these cases, $\vdash A \to \Box A$ remains true.

ment is incontrovertibly valid, so any disagreement must emanate from challenging their assumptions—assumptions which strike me as extremely plausible.

As it turns out, these seemingly innocuous assumptions have surprising implications. In particular, they entail that counterfactual conditional collapses into the material conditional; within the language of pure mathematics, '$A \to B$' holds just in case '$A \mathrel{\Box\!\!\to} B$' holds. The derivation of the collapse is as follows:

1. $A \to B, A \vdash B$ — *Modus Ponens*
2. $A \to B \vdash A \mathrel{\Box\!\!\to} B$ — *1, Counterfactual Deduction*
3. $\vdash (A \to B) \to (A \mathrel{\Box\!\!\to} B)$ — *2, Deduction Theorem*
4. $A \mathrel{\Box\!\!\to} B, A \vdash B$ — *Modus Ponens*
5. $A \mathrel{\Box\!\!\to} B \vdash A \to B$ — *4, Deduction Theorem*
6. $\vdash (A \mathrel{\Box\!\!\to} B) \to (A \to B)$ — *5, Deduction Theorem*
7. $\vdash (A \to B) \leftrightarrow (A \mathrel{\Box\!\!\to} B)$ — *3, 6 Cut and Classical Consequence*

For example, it is provable that 'If $2+2 = 4$, then $2+3 = 5$' obtains if and only if 'If it were the case that $2+2 = 4$, then $2+3$ would equal 5' obtains. While this particular example is seemingly unproblematic, the collapse has undesirable implications. In particular, it forces our hand on a contentious debate between the following three principles of counterfactual logic:

SUBSTITUTION OF EQUIVALENTS	If A is logically equivalent to B, then if $A \mathrel{\Box\!\!\to} C$ then $B \mathrel{\Box\!\!\to} C$.
SIMPLIFICATION	If $(A \vee B) \mathrel{\Box\!\!\to} C$ then $A \mathrel{\Box\!\!\to} C$ and $B \mathrel{\Box\!\!\to} C$.
FAILURE OF COUNTERFACTUAL STRENGTHENING	It is not the case that $A \mathrel{\Box\!\!\to} C$ entails $(A \wedge B) \mathrel{\Box\!\!\to} C$.

Each of these principles has received some measure of support. The substitution of equivalents is often defended on theoretical grounds. If two sentences are logically equivalent, it is difficult to see how any difference between them could affect the truth-values of counterfactuals they occur within. After all, they hold in precisely the same possible situations. Additionally, it is an immediate consequence of the Stalnaker/Lewis semantics for counterfactual conditionals that the Substitution of Equivalents holds. The closest possible worlds in which a sentence obtains are invariably the closest possible worlds in which equivalent sentences obtain, so accounts that rely upon the closeness of worlds do not distinguish between equivalent expressions. Even when the commitment to a particular semantics for counterfactual conditionals is dropped, many endorse a principle allowing for the substitution of equivalent expressions.[9]

Simplification is often defended by appeal to ordinary reasoning.[10] It would be strange to assert 'If Jack or Jill were to come to the party then the party would be fun, and if Jack were to come to the party, it would not be fun.' Similarly, it seems reasonable for someone to deny 'If it were to rain or not to rain, then the street would be wet' on the grounds that they deny 'If it were not to rain, then the street would be wet.' Both of these involve appeals to Simplification.

Similarly, the failure of counterfactual strengthening is often defended by appeal to the intuitive consistency of Sobel sequences.[11] It may be that if Tim were to take the aspirin, he would be fine, but if Tim were to take the aspirin and the cyanide, he would not be fine, and it may be that if the Federal Reserve were to lower the interest rate, the economy would grow, but if the Federal Reserve were to lower the interest rate and the European markets were to collapse, the economy would not grow. If these sentences are consistent, as they naturally seem to be, then Counterfactual Strengthening fails at least some of the time. Notably, this is a respect in which the counterfactual conditional appears to diverge from the material conditional. It is straightforward to establish that the material analog of counterfactual strengthening universally holds; that is, if $A \rightarrow C$ then $(A \wedge B) \rightarrow C$.

Despite these three principles' initial appeal, one must be abandoned—they are mutually inconsistent. The conflict between them is brought out in the following way:

1. $A \mathrel{\Box\!\!\rightarrow} C$ — *Supposition*
2. $A \vee (A \wedge B) \mathrel{\Box\!\!\rightarrow} C$ — *1, Substitution of Equivalents*
3. $(A \wedge B) \mathrel{\Box\!\!\rightarrow} C$ — *2, Simplification*

If Substitution of Equivalents and Simplification are both true, it follows that counterfactual strengthening universally succeeds. The two collectively entail that if 'If Sarah were to work hard, she would get a raise' is true, then 'If Sarah were to work hard and

[9]For an extended discussion of how substitution coheres with natural-language modals, see Kratzer (1981*a*,*b*, 1986, 1991).

[10]This was independently noticed by Fine (1975) and Nute (1975) in response to Lewis (1973*a*). For a response to Nute, see Loewer (1976), and for the ensuing discussion about disjunctive antecedents in counterfactual conditionals more generally, see Lewis (1977); Nute (1980); Alonso-Ovalle (2006).

[11]See Sobel (1970). For canonical discussions of Sobel sequences, see Stalnaker (1968); Lewis (1973*a*,*b*).

slap her boss, she would get a raise' is true as well.

While it is indisputable that these principles are incompatible, what we ought to do in light of this incompatibility is a matter of heated debate. Arguably, the most popular option is to retain the Substitution of Equivalents and the Failure of Counterfactual Strengthening, and to abandon Simplification. This option is forced upon us by the Stalnaker/Lewis semantics. As previously mentioned, this semantics licenses the Substitution of Equivalents, because equivalent expressions are true in the same possible situations. It also provides an intuitive explanation for the Failure of Counterfactual Strengthening. It may be that the closest worlds in which Sarah works hard are ones in which she gets a raise, but the closest worlds in which Sarah both works hard and slaps her boss are not ones in which she gets a raise, because the closest worlds in which she works hard are not ones in which she slaps her boss. Simplification fails when only one disjunct is relevant to the most-similar possible worlds. Perhaps all of the closest worlds in which either Jack or Jill come to the party are ones in which Jill comes to the party. In this case, the closest worlds in which Jack comes to the party are not relevant in determining the truth-value of 'If Jack or Jill were to come to the party, then the party would be fun.' Admittedly, abandoning Simplification is a theoretical cost, but the pertinent cases can arguably be accommodated pragmatically, rather than semantically.[12]

Others disagree. Recently, Fine (2012) provided a hyperintensional semantics for counterfactual conditionals—one which preserves both Simplification and the Failure of Counterfactual Strengthening and abandons the Substitution of Equivalents. Santorio (2018) advocates abandoning both the Substitution of Equivalents and Simplification, but preserves the Failure of Counterfactual Strengthening. And Kocurek (Forthcoming) provides independent reasons to abandon the Substitution of Equivalents. All counterpossibles (counterfactual conditionals with impossible antecedents) have equivalent antecedents, and few license the substitution of any impossible antecedent with another. If substitution principles fail for counterpossibles, it is reasonable to expect them to fail for ordinary counterfactuals as well. Debate rages on. While the Stalnaker/Lewis line remains prominent (minimally, given the enduring popularity of this semantics, it is an option many are tacitly committed to), it is safe to say that the fact that it forces our hand on this debate counts among its most controversial implications.

As it turns out, Yli-Vakkuri and Hawthorne's assumptions also force our hand in this debate, *but force it differently than Stalnaker and Lewis do*. Due to the collapse of the counterfactual conditional to the material conditional, their assumptions entail that the Substitution of Equivalents and Simplification are both true. Consequently, these assumptions entail that Counterfactual Strengthening universally succeeds.

The derivation of the Substitution of Equivalents is as follows:

Suppose that A is logically equivalent to B.

[12]See, e.g., Klinedinst (2009).

1. $B \vdash A$ *Classical Consequence*

2. $A \mathrel{\Box\!\!\rightarrow} C, A \vdash C$ *Modus Ponens*

3. $A \mathrel{\Box\!\!\rightarrow} C, B \vdash C$ *1, 2 and Cut*

4. $A \mathrel{\Box\!\!\rightarrow} C \vdash B \mathrel{\Box\!\!\rightarrow} C$ *3, Counterfactual Deduction*

5. $\vdash (A \mathrel{\Box\!\!\rightarrow} C) \rightarrow (B \mathrel{\Box\!\!\rightarrow} C)$ *4, Deduction Theorem*

The derivation of Simplification is as follows:

6. $A \vdash A \vee B$ *Classical Consequence*

7. $(A \vee B) \mathrel{\Box\!\!\rightarrow} C, A \vee B \vdash C$ *Modus Ponens*

8. $(A \vee B) \mathrel{\Box\!\!\rightarrow} C, A \vdash C$ *6, 7 and Cut*

9. $(A \vee B) \mathrel{\Box\!\!\rightarrow} C, \vdash A \mathrel{\Box\!\!\rightarrow} C$ *8, Counterfactual Deduction*

10. $\vdash ((A \vee B) \mathrel{\Box\!\!\rightarrow} C) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C)$ *9, Deduction Theorem*

As with the derivation of the necessity of mathematics, it not only follows that the Substitution of Equivalents and Simplification hold, but it is always provable that they hold. As we have already seen, these principles, in turn, entail the success of Counterfactual Strengthening. Therefore, Yli-Vakkuri and Hawthorne's assumptions entail that every instance of Counterfactual Strengthening expressible in the language of pure mathematics succeeds.

How worrying is this result? Presumably, this (at least partially) depends on what is expressible in the language of pure mathematics. If Sobel sequences are expressible, then these assumptions have untenable implications. Few react to the conflict between Substitution, Simplification and Strengthening by jettisoning Strengthening; the plausibility of Sobel sequences seems indispensable to modal reasoning. And so, determining the viability of their program requires an account of what constitutes pure mathematics. Without one, it is impossible to determine whether Sobel sequences can be expressed. It would be question-begging to identify pure mathematics with those branches that are necessarily true—some other characterization is required.

Yli-Vakkuri and Hawthorne do not specify the boundaries of pure mathematics. It is partially for this reason that the present worry is merely a worry. Perhaps their view could be supplemented by an account of pure mathematics—one that evades the problems Counterfactual Strengthening generates. However, there is some textual evidence that their conception of pure mathematics is susceptible to this concern. This arises in response to a potential objection they consider. Some might suggest that mathematical counterfactuals are dispensable to mathematics. Mathematicians may employ them in order to improve readability or to add linguistic flair, but they could be removed without affecting any substantive result. If mathematical counterfactuals are dispensable, we ought not derive substantive modal conclusions from them. This worry is compounded by the collapse from the counterfactual to the material conditional; in every case where mathematicians employ a counterfactual conditional, they could employ the material conditional instead.

Yli-Vakkuri and Hawthorne deny that mathematical counterfactuals are dispensable, claiming the following:

> Counterfactuals are absolutely indispensable to what mathematics contributes to our total body of knowledge...Note first that myriad applications of mathematics to the hustle and bustle of both everyday life and engineering require our knowing that mathematical truths would remain true even if things had gone differently in various ways. For example, in justifying a particular engineering solution, one often appeals to mathematical truths in reasoning about how things would have gone if one had opted for an alternative solution. In doing so one assumes—and if one is successful, one knows—that those mathematical truths would have been true even if one had opted for the alternative solution. Note second that, as the queen of the sciences, mathematics is primed for application in any area of objective inquiry, whether it be the science of electromagnetism, the theory of rook and pawn endings, or natural language semantics. (Pg. 14)

This strongly suggests that genuinely mathematical counterfactuals occur in disciplines ranging from engineering to electromagnetism to natural language semantics. After all, if the language of mathematics is incapable of expressing these counterfactuals, how could they lend support for the dispensability of counterfactuals in mathematics? Sobel Sequences are derivable in every discipline they mention. An engineer might derive the fact that, if a pulley were to double in size, it could lift a heavy box, but would deny that if a pulley were to double in size and be made of twine, it could lift a heavy box. A physicist might conclude that if an electron were to be placed in a field, it would accelerate, but deny that if an electron were to be placed in a field and an equal-but-opposite force were to be introduced, it would accelerate. Both the engineer and physicist thus deny the felicity of Counterfactual Strengthening in the counterfactuals they appeal to. Yli-Vakkuri and

Hawthorne's assumptions, which entail that counterfactual strengthening succeeds, are at odds with this practice. Of course, they might deny that these sentences can be expressed within the language of mathematics, but we then return to the puzzle about what the language of mathematics can express.

I myself doubt that there is a neat division between pure and impure mathematics. Of course, some disciplines are more applied than others, but I see no reason to think that there is a sharp divide between the pure and impure fields. Yet without such a distinction, we cannot determine whether Sobel sequences can be expressed. Determining the plausibility of Yli-Vakkuri and Hawthorne's assumptions thus depends on where this demarcation lies, as it determines whether counterfactual strengthening has tenable results.

Before turning to my second worry, there is one further concern about the intelligibility of mathematical counterfactuals on Yli-Vakkuri and Hawthorne's approach. It is unclear what the semantics underlying these counterfactuals is. The Stalnaker/Lewis semantics requires Simplification to be false, while Yli-Vakkuri and Hawthorne's assumptions entail that Simplification is true. This semantics remains the dominant interpretation of counterfactuals in philosophy (and beyond). Without the ability to appeal to this semantics, something else ought to take its place. And while there are alternate semantics for counterfactual conditionals available, I know of none amenable to Yli-Vakkuri and Hawthorne's logic. In order for their axioms to rest on solid foundations, a semantics compatible with their system ought to be provided. Currently available options are unsuitable.

A Worry Concerning Counterfactual Metalogic

My second worry could be framed in several ways, perhaps the most charitable of which is this: Yli-Vakkuri and Hawthorne defend the wrong thing. The only axiom that receives sustained defense is Counterfactual Deduction. From a dialectic perspective, this is entirely understandable; it is their most novel assumption. However, it performs extraordinarily little theoretical work in deriving the necessity of mathematics, and other axioms function equally well in its place. Instead, the axiom which predominantly responsible for their result, and which carries the most weight in establishing the commitment to an **S5** modality, is Deduction Theorem. Without a compelling reason to accept Deduction Theorem in this context, we remain without a compelling argument for the necessity of mathematics.

What does the claim that Counterfactual Deduction performs less theoretical work than Deduction Theorem performs amount to? After all, each is strictly needed in the derivation, and it would be silly to measure an axiom's importance by its number of occurrences within a proof.[13] I suspect that the relative impotence of Counterfactual

[13]More cautiously, it would be silly to measure an axiom's importance by its number of occurrences in the type of system Yli-Vakkuri and Hawthorne defend. It is a more plausible gauge in a system like Linear Logic, according to which axioms can be thought of as resources which can only be used a specific number of times.

Deduction is brought out most perspicuously by comparing Yli-Vakkuri and Hawthorne's system to other modal logics. As it turns out, their system is probably equivalent to appending Deduction Theorem to a **T** modal logic. Everything provable in their system is provable by combining **T** with Deduction Theorem, and everything not provable in their system is not provable by combining Deduction Theorem with **T**. The two are alternate axiomatizations of the same logic.

I establish this indirectly. What I immediately prove is that their system is equivalent to combining Deduction Theory to a system of counterfactual logic formalized by Williamson (2007).[14] Each set of axioms can be used to derive the other. Independently, Williamson proves that his system is equivalent to **T**; it is an immediate consequence that their logic is equivalent to appending Deduction Theorem to **T**. Williamson's axioms are the following:

PC	If A is a truth-functional tautology then $\vdash A$
REFLEXIVITY	$\vdash A \mathrel{\Box\!\!\rightarrow} A$
VACUITY	$\vdash (\neg A \mathrel{\Box\!\!\rightarrow} A) \rightarrow (B \mathrel{\Box\!\!\rightarrow} A)$
MP	If $\vdash A \rightarrow B$ and $\vdash A$ then $\vdash B$
MP$\Box$	$\vdash (A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \rightarrow B)$
CLOSURE	If $\vdash B \rightarrow C$ then $\vdash (A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C)$
EQUIVALENCE	If A is equivalent to $A*$ then $\vdash A \mathrel{\Box\!\!\rightarrow} B$ iff $\vdash A* \mathrel{\Box\!\!\rightarrow} B$

Williamson also demonstrates that the fragment of this system without *MP*$\Box$ (which is sometimes referred to as 'weak strengthening') is equivalent to **K**. *MP*$\Box$ serves solely to derive the **T** axiom.

This equivalence is philosophically significant for several reasons. Yli-Vakkuri and Hawthorne could have relied on other axioms to achieve the same result. Replacing the bulk of their assumptions (including Counterfactual Deduction) with Williamson's would work just as well—as would postulating that the language of mathematics is capable of expressing modal claims at least as strong as **T**. However, while there is some degree of flexibility in the choice of axioms, one which remains constant throughout is Deduction Theorem. Regardless of which system is selected, it must also be assumed that Deduction Theorem obtains in order to derive the necessity of mathematics. It is partially, although not entirely, for this reason that I claim that Deduction Theorem requires defense.

Deduction Theorem is extraordinarily intuitive and is an immediate metatheorem of propositional and first-order logic.[15] It holds in many nonclassical systems as well.

See Girard (1987, 1998) for the introduction of this type of system.

[14]See Appendix for the details of this proof.

[15]Kleene (1952) states that Deduction Theorem was first proved for propositional and first-order logic by Herbrand (1930 (1971)).

However, many maintain that Deduction Theorem fails for modal logic.[16] For example, in *Logic for Philosophy*, Sider states:

> "What's more, even though $P \vdash \Box P$, it's *not* the case that $\vdash P \rightarrow \Box P$. (We'll be able to demonstrate this once we've proved soundness for K.) So the Deduction Theorem...fails for our axiomatic system K—and indeed, for all the axiomatic modal systems we will consider." (Sider, 2010, pg. 160—emphasis original).

Sider is far from an outlier in this regard; others who deny that Deduction Theorem obtains for modal logic include, but are not limited to, Smorynski (1984); Fagin et al. (1995); Chagrov and Zakharyaschev (1997); Ganguli and Nerode (2004); Fitting (2007). Typically, such philosophers do not maintain that the success of Deduction Theorem depends on the modal logic under consideration. Rather, they deny the admissibility of Deduction Theorem in any system which is at least as strong as **K**. Those who, in contrast, endorse Deduction Theorem for modal logic owe an explanation for why they maintain it succeeds.

Perhaps what is even more telling than the fact that many dispute Deduction Theorem is the reason they provide for doing so. This was suggested, to some extent, in Sider, but is stated more explicitly in Fitting (2007):

> "Modal logic raises problems for the notion of deduction. Suppose we want to show $X \rightarrow Y$ in some modal axiom system by deriving Y from X. So we add X to our axioms. Say, to make things both concrete and intuitive, that X is 'it is raining' and Y is 'it is necessarily raining.' Since X has been added to the axiom list the necessitation rule applies, and from X we conclude $\Box X$, that is Y. Then the Deduction Theorem would allow us to conclude that if it is raining, it is necessarily raining. This does not seem right—nothing would ever be contingent."

Fitting notes that one of the **K** axioms is necessitation—the claim that if A is a theorem then $\Box A$ is a theorem as well.[17] This assumption is remarkably uncontroversial; it allows one, for example, to conclude that the law of excluded middle necessarily obtains from the fact that it is provable that it actually obtains. However, when combined with Deduction Theorem it has an untenable result: there are no contingent truths. For any sentence A

[16]Debates over the validity of Deduction Theorem for modal logic have occurred since the formalization of modal logic. See Barcan (1946); Barcan Marcus (1953); Feys (1965) for examples of early philosophers who dispute Deduction Theorem for at least some modal systems. These early debates predominantly concern different problems than the one I primarily address. I simply note that, if these arguments succeed, then there are additional reasons to doubt Yli-Vakkuri and Hawthorne's assumptions.

[17]Note that this is not the only type of argument against Deduction Theorem for modal logic in the literature. In addition to the worries of the early modal logicians, Basin, Matthews and Viganó (1998) provide a semantic argument against Deduction Theorem which is largely independent from Fitting's concerns.

in the language of modal logic, $\vdash A \to \Box A$. On a standard Hamblin-style system, there is no distinction between a proposition functioning as a premise and functioning as an axiom. Once a proposition is added to the set of axioms, the necessitation rule applies and one may conclude that it is necessarily true. Deduction Theorem then entails that if that sentences is true then it is necessarily true. Because this holds for any proposition whatsoever, Deduction Theorem entails that all truths are necessary. This is not restricted to the language of pure mathematics, but applies to any claim expressible in modal logic.

At the outset, I charitably framed my worry as the need to defend Deduction Theorem. A less charitable framing is this: Yli-Vakkuri and Hawthorne's argument merely amounts to the re-discovery that Deduction Theorem and Necessitation jointly entail that every truth is necessarily true. And it is precisely this entailment that leads many to reject Deduction Theorem for modal logic.

I do not mean to suggest that Deduction Theorem is indefensible—perhaps it lies at the heart of the distinction between the language of pure mathematics and other modal languages. Rather, my point is that Deduction Theorem cries out for defense. It is neither new nor surprising that it can be employed to derive necessitation, what would be new is an argument that it is admissible in this context. Until such an argument is provided, we lack a compelling reason to accept this counterfactual logic as providing the foundation for the necessity of mathematics.

Admittedly, some logicians have attempted to resurrect forms of Deduction Theorem for modal logic. Yli-Vakkuri and Hawthorne might appeal to these defenses in support of their claim. The remainder of this section concerns what is, to my mind, the most successful such proposal by Hakli and Negri (2012). Broadly, I maintain that Yli-Vakkuri and Hawthorne cannot appeal to this version of Deduction Theorem without begging the question, as it is admissible only on the assumption that all mathematical truths necessarily obtain.

Hakli and Negri distinguish a notion of entailment in terms of truth from a notion in terms of validity. This distinction was noted independently by Avron (1991) in connection to first-order logic, who defines it as follows:

TRUTH: $\Gamma \vdash_t A$ iff every assignment in a first-order structure which makes Γ true also makes A true.

VALIDITY: $\Gamma \vdash_v A$ iff if Γ is valid (i.e., true on all assignments), then A is true.

Notably, while $A(x) \vdash_v \forall x F(x)$, $A(x) \nvdash_t \forall x F(x)$. On some assignments $A(x)$ is true while $\forall x F(x)$ is false, but if it is the case that $A(x)$ is true on all assignments, then $\forall x F(x)$ is as well. Halki and Negri offer a similar distinction for modal logic:

TRUTH: $\Gamma \vdash_t A$ iff given a frame and a valuation in that frame and a world in it, if Γ are true in that world then A is true in that world.

VALIDITY: $\Gamma \vdash_v A$ iff given a frame if Γ are true in every world, then A is true.

It should be no surprise that $\Box$ functions analogously to $\forall$—the terms function similarly in many logical respects. In particular, the parallel result to Avron's holds: while $A \vdash_v \Box A$, $A \nvdash_t \Box A$. That is to say, if A holds in every world, then $\Box A$ holds as well, but it may be that A holds in one world without holding in all. Hakli and Negri defend a version of Deduction Theorem restricted to $\vdash_v$. The necessitation result is unproblematic for $\vdash_v$, as it is a notion of entailment which already assumes that the results of its theorems necessarily obtain. However, Hakli and Negri deny Deduction Theorem for $\vdash_t$ in order to allow for contingent truths.

I take no issue with Hakli and Negri's distinction, nor do I dispute that Deduction Theorem holds for $\vdash_v$ in propositional modal logic. Yli-Vakkuri and Hawthorne might appeal to this notion of entailment—arguing that the notion of informal provability corresponds to $\vdash_v$, rather than $\vdash_t$. If this were so, then their use of Deduction Theorem in modal contexts may be justified.

There are several reasons to be wary of this suggestion. The notion of informal provability is intended to reflect mathematical practice. If it corresponds to $\vdash_v$, rather than $\vdash_t$, then there ought to be textual evidence that this is the case—evidence similar to that which supported Counterfactual Deduction. Until such evidence is provided, it would be unwarranted to assume that mathematicians operate with one type of entailment rather than the other. But the larger concern that an appeal to $\vdash_v$ itself presupposes the necessity of mathematics. For, given an arbitrary sentence A within the language of mathematics, it is the case that $A \vdash_v \Box A$—and Deduction Theorem then entails $\vdash_v A \rightarrow \Box A$. No further assumptions are required to establish the necessity of all mathematical claims; Counterfactual Deduction, Modus Ponens, and Cut are superfluous. The necessity of mathematics, on this assumption, is effectively built into $\vdash_v$ and Deduction Theorem. This project cannot be resurrected by a mere appeal to $\vdash_v$, for this appeal relies upon the assumption that the sentences within the language of pure mathematics are necessarily true. It would be question-begging in the extreme to justify the necessity of mathematics in this way—some other defense is required.

Deduction Theorem remains extremely controversial within modal logic. Many dispute its validity for propositional modal logic because it gives rise to necessitation; the claim that if sentence A is true then A is necessarily true. Given that this is precisely what Yli-Vakkuri and Hawthorne intend to demonstrate about the language of pure mathematics, a sustained defense is needed. And while Hakli and Negri offer a form of Deduction Theorem that succeeds, it cannot be assumed without argument to be the notion of informal provability in mathematics.

Conclusion

At the outset, I noted a tension between the two worries I raise: that, while the first could be understand as the claim that Yli-Vakkuri and Hawthorne's assumptions have implausible implications about counterfactual logic, the second is that the bulk of these assumptions perform minimal theoretical work. I do not believe these concerns are at odds. This is because while many of their assumptions are innocuous, Deduction Theorem is not. It is no surprise that it arises in both the derivations of the Substitution of Equivalents and Simplification; it carries weight both in deriving the necessity of mathematics as well as the controversial implications that Yli-Vakkuri and Hawthorne's theory has.

These worries are inconclusive; what is needed is a substantive defense where, now, there is none. However, I close by discussing an alternate path forward. I believe that there is another route to establishing the necessity of mathematics than that suggested by Yli-Vakkuri and Hawthorne. The assumptions it relies upon are controversial—perhaps nearly as controversial as theirs. But it is an avenue for further research should some be dismayed by Yli-Vakkuri and Hawthorne's prospects.

There has been resurgent interest in a theory of truthmaking: of the claim that there exists something on the side of the world—perhaps a state of affairs or an object—that makes something true on the side of language—perhaps a sentence or a proposition. This is a notion which can trace its routes back to Armstrong (1989, 2000, 2004) (if not before), but has witnessed renewed discussion due to the development of truth maker semantics, primarily by Fine (2013, 2014, 2016, forthcoming). This interest, however, has been met with some resistance. Most importantly, for the present discussion, is an argument against truthmakers present in Williamson (2013).

Williamson's argument against truthmaking precedes from his assumption that everything necessarily exists; if an object *o* exists, then *o* necessarily exists. For some objects, it is contingent whether or not they are concrete. If Socrates had not been born he would have existed as a merely possible child of his parents, but he would have existed nonetheless. This argument is based almost entirely on logical grounds; it is easier to formalize quantified modal logic with the assumption that everything necessarily exists in place.

Williamson's argument against truthmaking is that a standard assumption about truthmaking entails that there are no contingent truths; everything necessarily obtains. The assumption at issue is that truthmakers necessitate that which they make true; for example if the state of affairs of grass being green is a truthmaker for 'Grass is green,' then in any possible situation in which there is a state of grass being green, the sentence 'Grass is green' is true.[18] If everything necessarily exists, then every truthmaker necessarily exist. And if these truthmakers necessitate that which they are truthmakers of, then every sentence with a truthmaker is necessarily true. If, for example, the state of affairs of grass being

[18]Armstrong's characterization of truthmaking was specified in modal terms. A truthmaker *t* made a sentence *S* true just in case it necessitated the truth of *S*. However, those who do not analyze truthmaking in directly modal terms often also maintain that truthmakers necessitate that which they make true.

green necessarily exists and necessitates the truth of 'grass is green' then it is necessary that grass is green. But this, Williamson claims, is absurd, and a theory of truthmaking ought to be rejected.

For my part, I am not convinced that a theory of truthmaking is what ought to go; I am much happier to abandon the claim that everything necessarily exists. And I do not find it much more implausible to suggest that it is necessary that grass is green than it is to suggest that it is necessary that Socrates exists. Setting these concerns aside, this argument suggests a new route towards the necessity of mathematics. If truthmakers necessitate that which they make true, and if the truthmakers of mathematical claims necessarily exist, then sentences within the language of mathematics are necessarily true.

This argument contains two premises, each of which requires independent defense. The first—that truthmakers necessitate what is necessarily true—is not particular to mathematics, and presumably piggybacks on standard defenses of truthmaking. But the second—that the truthmakers of mathematical claims necessarily exist—requires particular support. It demands some characterization of what it is that makes mathematical claims true, and that these things exist necessarily. One possibility (which admittedly may be restricted to arithmetic) is that the relevant truthmakers are *sets*.[19] Many maintain that sets exist in every possible world in which the objects contained within them exist.[20] If this is so, then $\varnothing$ necessarily exists, $\{\varnothing\}$ necessarily exists, etc. And because these sets necessitate the truth of arithmetic claims, arithmetic necessarily holds.

One might prefer other characterizations of the truthmakers of arithmetic claims. Although I am susceptible to the idea that these sets necessarily exist, and that they make arithmetic claims true, any view about the of truthmakers for mathematics will ensure the necessity of mathematics if these truthmakers necessarily exist. And so there is another route to the necessity of mathematics than the one Yli-Vakkuri and Hawthorne provide. One might attempt to reinforce their argument by rectifying the implausible implications of their assumptions and defending Deduction Theorem, but—alternatively—one might precede by examining what it is that makes mathematics true.

[19]Note that this differs slightly from the claim that numbers themselves are sets. It could be that numbers are sets, in which case the relations they stand in to each other are guaranteed by the existence of sets. But mathematical structuralists persuaded by Benacerraf (1965 (1983)) might still believe that sets are truthmakers for arithmetic, as sets guarantee that mathematical structures are instantiated.

[20]See, e.g., Fine (1994).

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Appendix

The following is a proof that the system of counterfactual logic developed by Yli-Vakkuri and Hawthorne (2018) is equivalent to appending Deduction Theorem to a **T** modal logic. This proof proceeds indirectly. What I immediately establish is that the system is equivalent to a counterfactual logic developed in Williamson (2007), when appended to Deduction Theorem. However, Williamson independently proves that his logic (without Deduction Theorem) is equivalent to **T**; it follows that Yli-Vakkuri and Hawthorne's system is equivalent to the conjunction of **T** with Deduction Theorem. The only additional assumption I make about Williamson's logic is that it is monotonic; i.e., that if $\Gamma \vdash B$ then $\Gamma, A \vdash B$.

I begin by establishing that Williamson's axioms follow from Yli-Vakkuri and Hawthorne's.

<u>PC:</u> If A is a truth-functional tautology, then $\vdash A$

This follows immediately from Classical Consequence; i.e., if A is a truth-functional tautology, then

$$\varnothing \vdash A \tag{1}$$

<u>REFLEXIVITY:</u> $\vdash A \mathrel{\Box\!\!\rightarrow} A$

Classical Consequence entails:

$$A \vdash A \tag{2}$$

(2) and Deduction Theorem then entail:

$$\varnothing \vdash A \mathrel{\Box\!\!\rightarrow} A \tag{3}$$

<u>VACUITY:</u> $\vdash (\neg A \mathrel{\Box\!\!\rightarrow} A) \rightarrow (B \mathrel{\Box\!\!\rightarrow} A)$

An instance of Modus Ponens is:

$$\neg A, \neg A \mathrel{\Box\!\!\rightarrow} A, B \vdash A \tag{4}$$

(4) and Deduction Theorem then entail:

$$\neg A \mathrel{\Box\!\!\rightarrow} A, B \vdash \neg A \rightarrow A \tag{5}$$

Classical Consequence entails:

$$\neg A \to A \vdash A \tag{6}$$

(5), (6), and Cut collectively entail:

$$\neg A \mathrel{\Box\!\!\rightarrow} A, B \vdash A \tag{7}$$

(7) and Counterfactual Deduction then entail:

$$\neg A \mathrel{\Box\!\!\rightarrow} A \vdash B \mathrel{\Box\!\!\rightarrow} A \tag{8}$$

And, finally, (8) and Deduction Theorem entail:

$$\varnothing \vdash (\neg A \mathrel{\Box\!\!\rightarrow} A) \to (B \mathrel{\Box\!\!\rightarrow} A) \tag{9}$$

<u>MP:</u> If $\vdash A \to B$ and $\vdash A$, then $\vdash B$.

Let us suppose the following:

$$\varnothing \vdash A \to B \tag{10}$$

$$\varnothing \vdash A \tag{11}$$

An instance of Modus Ponens—which is not to be confused with MP—is the following:

$$A \to B, A \vdash B \tag{12}$$

(10), (12) and Cut entail:

$$A \vdash B \tag{13}$$

(11), (13) and Cut then entail:

$$\varnothing \vdash B \tag{14}$$

MP$\Box$ (Weak Centering): $\vdash (A \Box\!\!\rightarrow B) \rightarrow (A \rightarrow B)$

An instance of Modus Ponens is the following:

$$A \Box\!\!\rightarrow B, A \vdash B \tag{15}$$

(15) and Deduction Theorem entail:

$$A \Box\!\!\rightarrow B \vdash A \rightarrow B \tag{16}$$

(16) and Deduction Theorem entail:

$$\varnothing \vdash (A \Box\!\!\rightarrow B) \rightarrow (A \rightarrow B) \tag{17}$$

CLOSURE: If $\vdash B \rightarrow C$ then $\vdash (A \Box\!\!\rightarrow B) \rightarrow (A \Box\!\!\rightarrow C)$

Let us suppose that:

$$\varnothing \vdash B \rightarrow C \tag{18}$$

An instance of Modus Ponens is the following:

$$B, B \rightarrow C \vdash C \tag{19}$$

(18), (19) and Cut entail:

$$B \vdash C \tag{20}$$

Another instance of Modus Ponens is:

$$A, A \Box\!\!\rightarrow B \vdash B \tag{21}$$

(20), (21) and Cut then entail:

$$A, A \Box\!\!\rightarrow B \vdash C \tag{22}$$

(22) and Counterfactual Deduction then entail:

$$A \mathrel{\Box\!\!\rightarrow} B \vdash A \mathrel{\Box\!\!\rightarrow} C \tag{23}$$

And, finally, (23) and Deduction Theorem entail:

$$\varnothing \vdash (A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C) \tag{24}$$

<u>EQUIVALENCE:</u> If A is equivalent to $A*$, then $\vdash A \mathrel{\Box\!\!\rightarrow} B$ iff $\vdash A* \mathrel{\Box\!\!\rightarrow} B$

Let us suppose that:

$$A \leftrightarrow A* \tag{25}$$

I begin by establishing that if $\vdash A \mathrel{\Box\!\!\rightarrow} B$, then $\vdash A* \mathrel{\Box\!\!\rightarrow} B$. Let us suppose that:

$$\varnothing \vdash A \mathrel{\Box\!\!\rightarrow} B \tag{26}$$

(26) and the Monotonicity entail:

$$A* \vdash A \mathrel{\Box\!\!\rightarrow} B \tag{27}$$

An instance of Modus Ponens is:

$$A*, A, A \mathrel{\Box\!\!\rightarrow} B \vdash B \tag{28}$$

(27), (28) and Cut entail:

$$A*, A \vdash B \tag{29}$$

(25) and Classical Consequence entail:

$$A* \vdash A \tag{30}$$

(29), (30) and Cut entail:

$$A* \vdash B \tag{31}$$

(31) and Counterfactual Deduction then entail:

$$\varnothing \vdash A* \mathrel{\Box\!\!\rightarrow} B \tag{32}$$

A parallel proof establishes that if $\vdash A* \mathrel{\Box\!\!\rightarrow} B$ then $\vdash A \mathrel{\Box\!\!\rightarrow} B$. From this it follows that:

$$\vdash A \mathrel{\Box\!\!\rightarrow} B \text{ iff } \vdash A* \mathrel{\Box\!\!\rightarrow} B \tag{33}$$

DEDUCTION THEOREM

Deduction Theorem is an axiom in both systems under consideration; it follows trivially from itself.

Therefore, all of Williamson's axioms (with Deduction Theorem) follow from Yli-Vakkuri and Hawthorne's. In order to establish the equivalence of these systems, it is sufficient to prove Yli-Vakkuri and Hawthorne's axioms from Williamson's (with Deduction Theorem). I precede slightly out of order from Williamson's presentation in order to use earlier proofs to facilitate later ones. For the purposes of this paper, it suffices to demonstrate the unary instance of Cut (which, incidentally, is the only instance employed in their demonstration of the necessity of mathematics).

CUT: If $\Gamma \vdash A$ and $\Pi, A \vdash B$ then $\Gamma, \Pi \vdash B$.

Let us suppose that:

$$\Gamma \vdash A \tag{34}$$

$$\Pi, A \vdash B \tag{35}$$

(35) and Monotonicity entail:

$$\Gamma, \Pi, A \vdash B \tag{36}$$

(36) and Deduction Theorem entail:

$$\Gamma, \Pi \vdash A \rightarrow B \tag{37}$$

(34) and Monotonicity entail:

$$\Gamma, \Pi \vdash A \tag{38}$$

(37), (38) and MP then entail:

$$\Gamma, \Pi \vdash B \tag{39}$$

CLASSICAL CONSEQUENCE: If A follows from Γ by classical logic, then $\Gamma \vdash A$

Let us suppose that:

$$A \text{ follows from } \Gamma \text{ by classical logic.} \tag{40}$$

(40) and PC entail:

$$\varnothing \vdash \Gamma \rightarrow A \tag{41}$$

(41) and the Monotonicity entail:

$$\Gamma \vdash \Gamma \rightarrow A \tag{42}$$

PC entails:

$$\Gamma \vdash \Gamma \tag{43}$$

(42), (43) and MP collectively entail:

$$\Gamma \vdash A \tag{44}$$

MODUS PONENS: $\Gamma, A \Rightarrow B, A \vdash B$ where $\Rightarrow$ is either the material or counterfactual conditional.

This proof precedes in two steps—one for the material conditional and the other for the counterfactual conditional. Let us begin with the material conditional. Classical Consequence, having been just established, entails:

$$\Gamma, A \rightarrow B, A \vdash A \tag{45}$$

In addition, Classical Consequence entails:

$$\Gamma, A \rightarrow B, A \vdash A \rightarrow B \tag{46}$$

(45), (46) and MP entail:

$$\Gamma, A \rightarrow B, A \vdash B \tag{47}$$

This establishes the material version of Modus Ponens. The counterfactual version requires additional steps. First, $MP\Box$ entails:[21]

$$\varnothing \vdash (A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \rightarrow B) \tag{48}$$

Due to Monotonicity, this entails:

$$\Gamma, A \mathrel{\Box\!\!\rightarrow} B, A \vdash (A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (A \rightarrow B) \tag{49}$$

Classical Consequence, in turn, entails:

$$\Gamma, A \mathrel{\Box\!\!\rightarrow} B, A \vdash A \mathrel{\Box\!\!\rightarrow} B \tag{50}$$

(49), (50) and MP collectively entail:

$$\Gamma, A \mathrel{\Box\!\!\rightarrow} B, A \vdash A \rightarrow B \tag{51}$$

Classical Consequence entails:

$$\Gamma, A \mathrel{\Box\!\!\rightarrow} B, A \vdash A \tag{52}$$

And, finally, (51), (52) and MP entail:

$$\Gamma, A \mathrel{\Box\!\!\rightarrow} B, A \vdash B \tag{53}$$

[21]Note that this is the only instance where MP□ is employed within this proof; it is not needed to prove any other axiom that Hawthorne and Yli-Vakkuri rely upon.

COUNTERFACTUAL DEDUCTION: If $\Gamma, A \vdash B$ then $\Gamma \vdash A \boxright B$

Let us suppose:

$$\Gamma, A \vdash B \tag{54}$$

(54) and Deduction Theorem entail:

$$\Gamma \vdash A \rightarrow B \tag{55}$$

(55) and Closure entail:

$$\Gamma \vdash (A \boxright A) \rightarrow (A \boxright B) \tag{56}$$

An application of Reflexivity is:

$$\varnothing \vdash A \boxright A \tag{57}$$

Due to Monotonicity, this entails:

$$\Gamma \vdash A \boxright A \tag{58}$$

(56), (58) and MP then entail:

$$\Gamma \vdash A \boxright B \tag{59}$$

DEDUCTION THEOREM

As before, Deduction Theorem is an axiom in both systems under consideration and follows trivially from itself.

Therefore, all Yli-Vakkuri and Howthorne's axioms follow from Williamson's when appended to Deduction Theorem. Because each set of axioms can be derived from the other, the two systems are equivalent. As I mentioned at the outset, Williamson independently established that his system is equivalent to a **T** modal logic. It follows that Yli-Vakkuri and Howthorne's axioms are equivalent to appending **T** to Deduction Theorem. Everything provable in their system is provable in the conjunction of Deduction Theorem with **T**; nothing which cannot be proven in their system can be proven in the conjunction of Deduction Theorem with **T**.