A Brief and Friendly Introduction to Mixed-Effects Models in Psycholinguistics

Cluster-specific parameters ("random effects")

Parameters governing inter-cluster variability

Shared parameters ("fixed effects")

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25 March 2009
Goals of this talk

- Briefly review generalized linear models and how to use them
- Give a precise description of multi-level models
- Show how to draw inferences using a multi-level model (fitting the model)
- Discuss how to interpret model parameter estimates
  - Fixed effects
  - Random effects
- Briefly discuss multi-level logit models
Reviewing generalized linear models I

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The picture:
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Reviewing generalized linear models I

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Assumptions of the generalized linear model (GLM):

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   \[
   \eta = \alpha + \beta_1 X_1 + \cdots + \beta_N X_N \quad \text{(linear predictor)}
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\eta = l(\mu) \quad \text{(link function)}
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4. There is some noise distribution of \( Y \) around the predicted mean \( \mu \) of \( Y \):

\[
P(Y = y; \mu)
\]
Linear regression, which underlies ANOVA, is a kind of generalized linear model.
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Linear regression, which underlies ANOVA, is a kind of generalized linear model.

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▶ Noise is normally (\(=\)Gaussian) distributed around 0 with standard deviation \(\sigma\):

\[ \epsilon \sim N(0, \sigma) \]

▶ This gives us the traditional linear regression equation:

\[ Y = \alpha + \beta_1 X_1 + \cdots + \beta_n X_n + \epsilon \]
Reviewing GLMs IV

- How do we fit the parameters $\beta_i$ and $\sigma$ (choose model coefficients)?
- There are two major approaches (deeply related, yet different) in widespread use:

\[ Y = \alpha + \beta_1 X_1 + \cdots + \beta_n X_n + \epsilon \]
How do we fit the parameters $\beta_i$ and $\sigma$ (choose model coefficients)?

There are two major approaches (deeply related, yet different) in widespread use:

- The principle of maximum likelihood: pick parameter values that maximize the probability of your data $Y$

  $\text{choose } \{\beta_i\} \text{ and } \sigma \text{ that make the likelihood } P(Y|\{\beta_i\}, \sigma) \text{ as large as possible}$
Reviewing GLMs IV

How do we fit the parameters $\beta_i$ and $\sigma$ (choose *model coefficients*)?

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- Bayesian inference: put a probability distribution on the model parameters and update it on the basis of what parameters best explain the data
  
  $$P(\{\beta_i\}, \sigma|Y) = \frac{P(Y|\{\beta_i\}, \sigma)P(\{\beta_i\}, \sigma)}{P(Y)}$$
Reviewing GLMs IV

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You are studying non-word RTs in a lexical-decision task
Reviewing GLMs V: a simple example

- You are studying non-word RTs in a lexical-decision task.
  *tpozt*  
  *Word or non-word?*

- Non-words with different neighborhood densities should have different average RT.*

- A simple model: assume that neighborhood density has a linear effect on average RT, and trial-level noise is normally distributed.*
  \[
  \text{RT}_i = \alpha + \beta x_i + \epsilon_i \sim N(0, \sigma) 
  \]

- We need to draw inferences about \(\alpha\), \(\beta\), and \(\sigma\).
  
- e.g., "Does neighborhood density affect RT?" → is \(\beta\) reliably non-zero?
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- If $x_i$ is neighborhood density, our simple model is

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RT_i = \alpha + \beta x_i + \epsilon_i
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We’ll use length-4 nonword data from (Bicknell et al., 2008) (thanks!), such as:

<table>
<thead>
<tr>
<th>Few neighbors</th>
<th>Many neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>gaty</td>
<td>lish</td>
</tr>
<tr>
<td>peme</td>
<td>pait</td>
</tr>
<tr>
<td>rixy</td>
<td>yine</td>
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We’ll use length-4 nonword data from (Bicknell et al., 2008) (thanks!), such as:

- **Few neighbors**: gaty, peme, rixy
- **Many neighbors**: lish, pait, yine

There’s a wide range of neighborhood density:

![Bar chart with frequency counts for number of neighbors]
Reviewing GLMs VII: maximum-likelihood model fitting

\[ RT_i = \alpha + \beta X_i + \epsilon_i \sim \mathcal{N}(0, \sigma) \]

- Here’s a translation of our simple model into R:
  \[ RT \sim 1 + x \]
Reviewing GLMs VII: maximum-likelihood model fitting

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Example of fitting via maximum likelihood: one subject from Bicknell et al. (2008)

```r
> m <- glm(RT ~ neighbors, d, family="gaussian")
> summary(m)
```

```
Estimate Std. Error t value Pr(>|t|)
(Intercept) 382.997  26.837 14.271  <2e-16 ***
neighbors   4.828   6.553  0.737    0.466
```

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> sqrt(summary(m)[["dispersion"]])
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[1] 107.2248
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\[ \hat{\alpha} \]

\[ \hat{\beta} \]

\[ \hat{\sigma} \]

\[ \epsilon_i \]

\[ \sim N(0,\sigma) \]
Reviewing GLMs VII: maximum-likelihood model fitting

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\begin{array}{lrrrr}
\text{Estimate} & \text{Std. Error} & \text{t value} & \text{Pr(>|t|)} \\
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\end{array}
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\[ \hat{\alpha}, \hat{\beta} \]
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\[ \hat{\alpha}, \hat{\beta}, \hat{\sigma} \]
Estimated coefficients are what underlies "best linear fit" plots.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>383.00</td>
</tr>
<tr>
<td>neighbors</td>
<td>4.83</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
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</tr>
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Reviewing GLMs IX: Bayesian model fitting

Alternative to maximum-likelihood: Bayesian model fitting

\[
P(\{\beta_i\}, \sigma | Y) = \frac{\text{Likelihood}}{P(Y)} \cdot \frac{\text{Prior}}{P(\{\beta_i\}, \sigma)}
\]

\[p_{MCMC} = 1 - \text{HPD confidence region}\]

\[\alpha\]

\[\beta\]

\[\text{Neighbors}\]

\[\text{Intercept}\]

\[< \alpha^, \beta^ >\]
Reviewing GLMs IX: Bayesian model fitting

\[
P(\{\beta_i\}, \sigma | Y) = \frac{P(Y|\{\beta_i\}, \sigma) P(\{\beta_i\}, \sigma)}{P(Y)}
\]

- Alternative to maximum-likelihood: Bayesian model fitting
- **Simple** (uniform, non-informative) prior: all combinations of \((\alpha, \beta, \sigma)\) equally probable

\[
\begin{array}{c}
\text{Neighbors} \\
\text{Intercept}
\end{array}
\]

\[
\begin{array}{c}
200 \\
300 \\
400 \\
500 \\
600
\end{array}
\]

\[
\begin{array}{c}
-10 \\
0 \\
5 \\
10 \\
20
\end{array}
\]

\[
\text{\textless} \hat{\alpha}, \hat{\beta} >
\]

\[
p_{MCMC} = 1 - \text{largest possible symmetric confidence interval wholly on one side of 0}
\]
Reviewing GLMs IX: Bayesian model fitting

\[ P(\{\beta_i\}, \sigma | Y) = \frac{\text{Likelihood}}{P(Y)} \times \frac{\text{Prior}}{P(\{\beta_i\}, \sigma)} \]

- Alternative to maximum-likelihood: Bayesian model fitting
- Simple (uniform, non-informative) prior: all combinations of \((\alpha, \beta, \sigma)\) equally probable
- Multiply by likelihood \(\rightarrow\) posterior probability distribution over \((\alpha, \beta, \sigma)\)

\[ p_{MCMC} = 0.46 \]

\(-10\) \(0\) \(5\) \(10\) \(20\)

\(-10\) \(200\) \(300\) \(400\) \(500\) \(600\)

Neighbors

Intercept

\(<\hat{\alpha}, \hat{\beta}>\)
Reviewing GLMs IX: Bayesian model fitting

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Reviewing GLMs IX: Bayesian model fitting

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- Multiply by likelihood \(\rightarrow\) posterior probability distribution over \((\alpha, \beta, \sigma)\)
- Bound the region of highest posterior probability containing 95% of probability density \(\rightarrow\) HPD confidence region

\[
P(\alpha \mid Y)
\]

\[
\hat{\alpha}, \hat{\beta}
\]

\[
\text{pMCMC (Baayen et al., 2008)} = 1 \text{ minus the largest possible symmetric confidence interval wholly on one side of } 0
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▶ Alternative to maximum-likelihood: Bayesian model fitting

▶ Simple (uniform, non-informative) prior: all combinations of \((\alpha, \beta, \sigma)\) equally probable

▶ Multiply by likelihood → posterior probability distribution over \((\alpha, \beta, \sigma)\)

▶ Bound the region of highest posterior probability containing 95% of probability density → HPD confidence region

▶ \(p_{\text{MCMC}}\) (Baayen et al., 2008) is 1 minus the largest possible symmetric confidence interval wholly on one side of 0
But of course experiments don't have just one participant
Different participants may have different idiosyncratic behavior
And items may have idiosyncratic properties too
We’d like to take these into account, and perhaps investigate them directly too.
This is what multi-level (hierarchical, mixed-effects) models are for!
Recap of the graphical picture of a single-level model:
Multi-level Models III: the new graphical picture

Cluster-specific parameters ("random effects")

Shared parameters ("fixed effects")

Parameters governing inter-cluster variability
Multi-level Models III: the new graphical picture

Cluster-specific parameters ("random effects")

Shared parameters ("fixed effects")

Parameters governing inter-cluster variability

\[ \Sigma_b \]

\[ \theta \]

\[ \Sigma_b \]

\[ \theta \]
Multi-level Models III: the new graphical picture

Cluster-specific parameters ("random effects")

\[ \theta \]

\[ \Sigma_b \]

\[ b_1 \]

\[ x_{11} \]

\[ y_{11} \]

\[ \ldots \]

\[ x_{1n_1} \]

\[ y_{1n_1} \]

\[ b_2 \]

\[ x_{21} \]

\[ y_{21} \]

\[ \ldots \]

\[ x_{2n_2} \]

\[ y_{2n_2} \]

\[ \ldots \]

\[ \ldots \]

\[ b_M \]

\[ x_{M1} \]

\[ x_{Mn_M} \]

\[ y_{M1} \]

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\[ \ldots \]

\[ \ldots \]
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An example of a multi-level model:

- Back to your lexical-decision experiment
  - tpozt  \textit{Word or non-word?}
  - houze  \textit{Word or non-word?}

- Non-words with different \textit{neighborhood densities} should have different average decision time
Multi-level Models IV

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- Additionally, different participants in your study may also have:
  - different overall decision speeds
  - differing sensitivity to neighborhood density
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  - differing sensitivity to neighborhood density

- You want to draw inferences about all these things at the same time
Once again we’ll assume for simplicity that the number of word neighbors $x$ has a linear effect on mean reading time, and that trial-level noise is normally distributed.
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Random effects, starting simple: let each participant \( i \) have idiosyncratic differences in reading speed

\[
RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \sim N(0,\sigma_b) \\
\text{Noise} \sim N(0,\sigma_\epsilon)
\]
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\[ \sim N(0,\sigma_b) \]

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In R, we’d write this relationship as

\[ RT \sim 1 + x + (1 \mid \text{participant}) \]
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Random effects, starting simple: let each participant $i$ have idiosyncratic differences in reading speed

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In R, we’d write this relationship as

$$RT \sim 1 + x + \text{(1 | participant)}$$

Once again we can leave off the $1$, and the noise term $\epsilon_{ij}$ is implicit.
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Random effects, starting simple: let each participant $i$ have idiosyncratic differences in reading speed

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Multi-level Models VI: simulating data

\[ RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \sim N(0, \sigma_b) \quad \text{Noise} \sim N(0, \sigma_\epsilon) \]

- One beauty of multi-level models is that you can simulate trial-level data
- This is invaluable for achieving deeper understanding of both your analysis and your data
Multi-level Models VI: simulating data

\[ RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \]

\[ \sim N(0,\sigma_b) \quad \text{Noise} \sim N(0,\sigma_e) \]

- One beauty of multi-level models is that you can simulate trial-level data.
- This is invaluable for achieving deeper understanding of both your analysis and your data.

```r
## simulate some data
> sigma.b <- 125  # inter-subject variation larger than intra-subject, inter-trial variation
> sigma.e <- 40
> alpha <- 500
> beta <- 12
> M <- 6  # number of participants
> n <- 50  # trials per participant
> b <- rnorm(M, 0, sigma.b)  # individual differences
> nneighbors <- rpois(M*n,3) + 1  # generate num. neighbors
> subj <- rep(1:M,n)
> RT <- alpha + beta * nneighbors + b[subj] + rnorm(M*n,0,sigma.e)  # simulate RTs!
```
Participant-level clustering is easily visible
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This reflects the fact that inter-participant variation (125ms) is larger than inter-trial variation (40ms).
Participant-level clustering is easily visible
This reflects the fact that inter-participant variation (125ms) is larger than inter-trial variation (40ms)
And the effects of neighborhood density are also visible
Statistical inference with multi-level models

\[ RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \]

\[ \sim N(0,\sigma_b) \quad \text{Noise} \sim N(0,\sigma_e) \]

▶ Thus far, we’ve just defined a model and used it to generate data

We psycholinguists are usually in the opposite situation. . .

We have data and we need to infer a model

▶ Specifically, the “fixed-effect” parameters \( \alpha \), \( \beta \), and \( \sigma_{\epsilon} \), plus the parameter governing inter-subject variation, \( \sigma_b \)

▶ e.g., hypothesis tests about effects of neighborhood density: can we reliably infer that \( \beta \) is \( \{ \text{non-zero, positive, . . .} \} \)?

▶ Fortunately, we can use the same principles as before to do this:

▶ The principle of maximum likelihood

▶ Or Bayesian inference
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- The principle of maximum likelihood
- Or Bayesian inference
Fitting a multi-level model using maximum likelihood

\[
RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \\
\sim N(0, \sigma_b) \\
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\]

\[
m \leftarrow \text{lmer}(\text{time} \sim \text{neighbors.centered} + \text{(1 | participant)}, \text{dat}, \text{REML=F})
\]

\[
\text{> print}(m, \text{corr=F})
\]

[...]

Random effects:

<table>
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<tr>
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<th>Name</th>
<th>Variance</th>
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<tbody>
<tr>
<td>participant</td>
<td>(Intercept)</td>
<td>4924.9</td>
<td>70.177</td>
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<td>Residual</td>
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Number of obs: 1760, groups: participant, 44

Fixed effects:

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Fitting a multi-level model using maximum likelihood

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- Inter-participant variability $\sigma_b$ is what’s new:
  - Variability in average RT in the population from which the participants were drawn has standard deviation 70.18ms
Inferences about cluster-level parameters

$$RT_{ij} = \alpha + \beta x_{ij} + \hat{b}_i + \hat{\epsilon}_{ij}$$

- What about the participants’ idiosyncracies themselves—the $b_i$?
Inferences about cluster-level parameters

\[ RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij} \]

\( b_i \sim N(0, \sigma_b) \)

\( \epsilon_{ij} \sim N(0, \sigma) \)

▶ What about the participants’ idiosyncrasies themselves—the \( b_i \)?

▶ We can also draw inferences about these—you may have heard about BLUPs
Inferences about cluster-level parameters

\[
RT_{ij} = \alpha + \beta x_{ij} + b_i + \epsilon_{ij}
\]

\( \sim N(0, \sigma_b) \) Noise \( \sim N(0, \sigma_\epsilon) \)

What about the participants’ idiosyncracies themselves—the \( b_i \)?

We can also draw inferences about these—you may have heard about **BLUPs**

To understand these: committing to fixed-effect and random-effect parameter estimates determines a conditional probability distribution on participant-specific effects:

\[
P(b_i | \hat{\alpha}, \hat{\beta}, \hat{\sigma}_b, \hat{\sigma}_\epsilon)
\]
Inferences about cluster-level parameters

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- The BLUPS are the **conditional modes** of \( b_i \)—the choices that maximize the above probability
The BLUP participant-specific “average” RTs for this dataset are black lines on the base of this graph.

The solid line is a guess at their distribution.

The dotted line is the distribution predicted by the model for the population from which the participants are drawn.

Reasonably close correspondence.
Inferences about cluster-level parameters II

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Participants may also have idiosyncratic sensitivities to neighborhood density.
Inference about cluster-level parameters III

- Participants may also have idiosyncratic sensitivities to *neighborhood density*
- Incorporate by adding cluster-level slopes into the model:

\[ RT_{ij} = \alpha + \beta x_{ij} + b_{1i} + b_{2i} x_{ij} + \epsilon_{ij} \sim N(0, \Sigma_b) \]

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\]

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In R (once again we can omit the 1’s):

\[
RT \sim 1 + x + (1 + x \mid \text{participant})
\]
Inference about cluster-level parameters III

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> lmer(RT ~ neighbors.centered +
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```
Random effects:
Groups Name Variance Std.Dev. Corr
participant (Intercept) 4928.625 70.2042
neighbors.centered 19.421 4.4069 -0.307
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```
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These three numbers jointly characterize \( \hat{\Sigma}_b \)
Inference about cluster-level parameters IV

Correlation visible in participant-specific BLUPs

\[ \hat{\Sigma} = \begin{pmatrix} 70 & -0.20 \\ -0.3097 & 4.41 \end{pmatrix} \]
Inference about cluster-level parameters IV

**Participants**

- Correlation visible in participant-specific BLUPs
- Participants who were faster overall also tend to be more affected by neighborhood density

\[ \hat{\Sigma} = \left(\begin{array}{ccc}
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\end{array}\right) \]
Inference about cluster-level parameters IV

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Bayesian inference for multilevel models

\[
P(\{\beta_i\}, \sigma_b, \sigma_\epsilon | Y) = \frac{P(Y|\{\beta_i\}, \sigma_b, \sigma_\epsilon) P(\{\beta_i\}, \sigma_b, \sigma_\epsilon)}{P(Y)}
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- We can also use Bayes’ rule to draw inferences about fixed effects
Bayesian inference for multilevel models

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- Computationally more challenging than with single-level regression;
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- We can also use Bayes’ rule to draw inferences about fixed effects.
- Computationally more challenging than with single-level regression; Markov-chain Monte Carlo (MCMC) sampling techniques allow us to approximate it.
Why do you care???

You may be asking yourself:

*Why did I come to this workshop? I could do everything you just did with an ANCOVA, treating participant as a random factor, or by looking at participant means.*
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- Yes, but there are several respects in which multi-level models go beyond AN(C)OVA:
Why do you care???

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- Yes, but there are several respects in which multi-level models go beyond AN(C)OVA:
  1. They handle *imbalanced datasets* just as well as balanced datasets
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3. You can cross cluster-level effects
   ▶ Every trial belongs to both a participant cluster and an item cluster
   ▶ You can build a single unified model for inferences from your data
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2. You can use non-linear linking functions (e.g., logit models for binary-choice data)
3. You can cross cluster-level effects
   ▶ Every trial belongs to both a participant cluster and an item cluster
   ▶ You can build a single unified model for inferences from your data
   ▶ ANOVA requires separate by-participants and by-items analyses (quasi-$F'$ is quite conservative)
Why did I come to this workshop? I could do everything you just did with an ANCOVA, treating participant as a random factor, or by looking at participant means.

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- Much psycholinguistic data is *categorical* rather than *continuous*:
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Random effects:

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<tr>
<th>Groups</th>
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<td>participant</td>
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Number of obs: 1760, groups: participant, 44

Fixed effects:

| Estimate | Std. Error | z value | Pr(>|z|) |
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| (Intercept) | 3.16310    | 0.18998 | 16.649 < 2e-16 *** |
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\( \hat{\alpha} \) — participants usually right
\( \hat{\beta} \) — effect small compared with inter-subject variation

\( \hat{\sigma}_b \) (note there is no \( \hat{\sigma}_e \) for logit models)
Multi-level models may seem strange and foreign
But all you really need to understand them is three basic things
  - Generalized linear models
  - The principle of maximum likelihood
  - Bayesian inference
As you will see in the rest of the workshop (and the conference...?), these models open up many new interesting doors!

